THE FAR FIELD OF WAVES FORMED BY A DIPOLE IN A STRATIFIED FLUID STREAM FLOWING AT THE CRITICAL VELOCITY

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The spatial problem of steady waves being generated during the flow around a dipole in a uniform inviscid, incompressible, stratified fluid flow of finite depth is considered in a linear formulation. Approximate semi-asymptotic solutions of analogous problems by numerical methods are known [1, 2] for given fluid density distributions over the depth. An exact solution in the form of the sum of single integrals for waves from a source is obtained in [3]. Recently a uniform asymptotic has been determined for the leading front domain of a separate mode for the stream velocity c greater than the propagation velocity of the n-th mode long waves c_n [4, 5]. This asymptotic is expressed for a fluid of finite depth in terms of Airy functions [4] and for an infinitely deep fluid by Fresnel integrals [5]. The method of constructing the complete asymptotic expansions of the solution [3] is described in [6] for $c < c_n$.

The asymptotic of the exact solution (in a linear formulation) of the problem under consideration is calculated in this paper for the critical stream velocity $c = c_n$, including the uniform asymptotic for the leading front domain.

Let the horizontal flow of an inviscid, incompressible fluid of depth H flow around a submerged point dipole oriented against the flow. The fluid density in the unperturbed state $\rho_0(z)$ depends on one vertical coordinate z and does not decrease with depth. In a linear formulation, the field of vertical fluid particle displacements $\zeta(x, y, z)$ generated by the dipole is described by the equation

$$D^2 \frac{\partial}{\partial z}\left(\rho_0 \frac{\partial}{\partial z}\zeta\right) + \rho_0 (N^2 + D^2)\Delta_2\zeta = Mc^{-1}D^2\left\{\delta(x)\delta(y)\frac{d}{dz}[\rho_0\delta(z+H_1)]\right\} \quad (1)$$

with the boundary conditions

$$\left(D^2\frac{\partial}{\partial z} - g\Delta_2\right)\zeta = 0 \quad (z=0), \ \zeta = 0 \quad (z=-H), \quad (2)$$

where $D = c\partial/\partial x$: $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$; x, y are horizontal coordinates; the fluid flows at the velocity c in the positive x direction; the dipole is at a point with the coordinates $(0, 0, -H_1)$; $N^2 = -g\rho_0^{-1}d\rho_0/dz$ is the square of the Brunt–Väisälä frequency, M is the magnitude of the dipole moment, g is the free fall acceleration, and $\delta(\cdot)$ is the delta function. For an infinite homogeneous fluid [7] the dipole yields the pattern for the flow around a sphere of radius $\sqrt[3]{M/2\pi c}$.

An exact solution of an analogous problem is obtained in [1] for waves from a point source. It can be shown that the corresponding solution of (1) and (2) has the form

$$\zeta = M(2\pi^2 c)^{-1}\rho_0(-H_1)\sum_{n=0}^{\infty}\zeta_n(x, y, z), \quad (3)$$

$$\zeta_n = \operatorname{Re}\int_{-\pi/2}^{\pi/2}\Psi_n(\theta; z, -H_1)\,G\left[-R\beta_n^{1/2}\cos(\theta-\gamma)\right]d\theta.$$

Here R, γ are polar coordinates of the horizontal (x, y) plane, $x = R\cos\gamma$; $y = R\sin\gamma$; $\Psi_n = W_n(z;\theta)\frac{d}{dz}W_n(-H_1;\theta)$; $\beta_n^{1/2}$ is the arithmetic branch of the root; β_n and W_n are eigenvalues $(\beta_0 > \beta_1 > \ldots)$ and normalized eigenfunctions $\left(\int_{-H}^{0}\rho_0 W_n^2 dz = 1\right)$ of the Sturm–Liouville problem $\frac{d}{dz}\left(\rho_0\frac{d}{dz}W\right) + \rho_0(N^2\lambda - \beta)W = 0 \quad (-H < z < 0)$, $\frac{d}{dz}W - g\lambda W = 0 \ (z = 0)$, $W = 0 \ (z = -H)$,

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$\lambda = (c \cos\theta)^{-2}$. The G(u) in (3) is the analytic continuation of the function $\varphi(u) = \int_0^\infty t(t^2+1)^{-1} e^{-ut} dt \ (\mathrm{Re}\, u > 0)$ in the complex plane of the variable u with the slit $(-\infty, 0]$. Let us describe the properties of the function G(u) briefly. Deduced from the definition is

$$G(-u) = G(u) + is\pi e^{isu}, \quad s = \mathrm{sign}\,(\arg u); \tag{4}$$

$$G(u) \sim -\sum_{m=1}^{\infty} (-1)^m (2m-1)!\, u^{-2m} \ \text{при}\ |u| \to \infty,\ |\arg u| < \pi. \tag{5}$$

Let us use the notation $F(u) = \frac{d}{du}[G(u) + \ln u]$, where the principal branch of the logarithm is taken. The real parts of F(u) and the sum [G(u) + ln u] vary continuously during passage through the slit and along the real u axis, $\mathrm{Re}\, F(0) = \pi/2$, $\mathrm{Re}[G(u) + \ln u]_{u=0} = -C_0$, and C_0 is the Euler constant. The functions G(u) and F(u) are also connected by the relationship $\frac{d}{du} F(u) = -G(u)$.

The properties of the dispersion dependences $\beta_n(\lambda)$ are described in detail in [3]. It is essential for the purposes of this paper that $\beta_n(\lambda)$ grow monotonically for $\lambda \geq 0$, tend to infinity for $\lambda \to \infty$, where

$$\frac{d\beta_n}{d\lambda} = g\rho_0 W_n^2|_{z=0} + \int_{-H}^{0} \rho_0 N^2 W_n^2 dz, \tag{6}$$

and have one simple zero $\lambda = \lambda_n$. The critical velocity c_n for the n-th mode waves is related to λ_n by the simple relationship $c_n = \lambda_n^{-1/2}$. In the critical case $c = c_n$ under consideration in this paper, the function $r_{n1}(\theta) = \beta_n^{1/2}(c^{-2}\cos^{-2}\theta)$ from (3) is even, positive for $\theta \neq 0$, and $r_{n1}(0) = 0$, $\frac{d}{d\theta} r_{n1}(\pm 0) = \pm \varkappa_n \left(\varkappa_n = c_n^{-1}\sqrt{\beta'_{n\lambda}(\lambda_n)}\right)$.

Let us analyze the contribution of the n-th mode to the far domain of the wave field (as $R \to \infty$, $\gamma_1 \leq \gamma \leq \pi$, γ_1 is a small positive number). First we make some remarks about the technical aspect of calculating the asymptotic expansion of the integral (3). Note that for $c = c_n$ the argument of the function $G(\cdot)$ in (3) takes on only real values. It follows from (4) and (5) that

$$\mathrm{Re}\, G(u) \sim \delta\pi \sin u - \sum_{m=1}^{\infty} (-1)^m (2m-1)!\, u^{-2m}, \tag{7}$$

where $\delta = 0$ for $u > 0$ and $\delta = 1$ for $u < 0$ when $\mathrm{Im}\, u = 0$ and $|u| \to \infty$. Extracting the neighborhood of the zeroes of the expression $\Delta_{n1}(\theta) = r_{n1}(\theta)\cos(\theta - \gamma)$, in conformity with (7) we obtain a Fourier integral and power series for (3) for the remaining part of the interval of integration. The asymptotic of the Fourier integral is calculated by the stationary phase method [8]. The contributions of the zeros Δ_{n1} are found by integration by parts [6].

The functions $\Delta_{n1}(\theta)$ have for $0 < \gamma < \pi$, $\gamma \neq \pi/2$ two simple roots $\theta_0 = 0$ and $\theta_1 = \gamma - \pi/2$, since

$$\frac{\partial}{\partial\theta}\Delta_{n1}(\theta_1) = r_{n1}(\theta_1) \text{ and } \frac{\partial}{\partial\theta}\Delta_{n1}(\pm 0) = \pm\varkappa_n \cos\gamma. \tag{8}$$

If $\gamma = 0$ or $\gamma = \pi$, then Δ_{n1} has just one zero θ_0 (it is known that $r_{n1}\cos\theta \to c^{-1}\max N(z)$ as $\theta \to \pi/2$, $n \geq 1$ and $r_{01}\cos\theta \to \infty$), while if $\gamma = \pi/2$, then Δ_{n1} has one multiple zero. Let us first examine the case when Δ_{n1} has two zeros $\theta_0 \neq \theta_1$. We select nonintersecting neighborhoods V_0 and V_1 of the points θ_0 and θ_1, respectively, and we arrange a partition of unity [8]

$$\eta_0(\theta) + \eta_1(\theta) + \eta_2(\theta) \equiv 1. \tag{9}$$

Here the functions $\eta_k(\theta)$ (k = 0,1) equal zero outside of V_k, are infinitely differentiable, $\eta_k(\theta_k) = 1$ and $d^m\eta_k(\theta_k)/d\theta^m = 0$ for $m \geq 1$, and the function $\eta_2(\theta)$ is defined by the identity (9). Now the expression (3) can be written in the form of the sum

$$\zeta_n = \sum_{k=0}^{2} \zeta_{nk}, \quad \zeta_{nk} = \mathrm{Re}\int_{-\pi/2}^{\pi/2} \Psi_{nk} G(-R\Delta_{n1})\, d\theta, \quad \Psi_{nk} = \Psi_n \eta_k. \tag{10}$$

Let us calculate the asymptotic as $R \to +\infty$ for each of the components of (10). Let us regularize the argument of the function $G(\cdot)$ by using the notation $r_n(\theta) = \mathrm{sign}(\theta) r_{n1}(\theta)$ and $\Delta_n = r_n \cos(\theta - \gamma)$. Using (4) we obtain

$$\zeta_{n0} = \pi \operatorname{Im} \int\limits_{V_0, \theta \leqslant 0} \Psi_{n0}^{-} e^{iR\Delta_n} d\theta + \operatorname{Re} \int\limits_{V_0} \Psi_{n0} G(-R\Delta_n)\, d\theta. \tag{11}$$

Within an accuracy of $O(R^{-\infty})$, the first component in (11) equals the contribution of the boundary point $\theta = 0$ [8]; the asymptotic of the second component and of ζ_{n1} is found by integration by parts [6]. Consequently

$$\zeta_{n0} = B_n(R, \gamma) + Z_{n0}(R, \gamma), \ \zeta_{n1} = Z_{n1}(R, \gamma), \tag{12}$$

$$B_n(R, \gamma) \sim -\pi \sum_{m=0}^{\infty} (-1)^m R^{-(2m+1)} M^{2m} \left[\frac{\Psi_n}{\Delta'_{n\theta}} \right]_{\theta=0},$$

$$Z_{nk}(R, \gamma) \sim \sum_{m=1}^{\infty} (-1)^m R^{-2m} \int\limits_{V_k} \ln|\Delta_n| \frac{d}{d\theta} M^{2m-1} \left[\frac{\Psi_{nk}}{\Delta'_{n\theta}} \right] d\theta, \ M = \frac{1}{\Delta'_{n\theta}} \frac{d}{d\theta}.$$

The principal term of the asymptotic is $\zeta_{n0} = -\frac{\pi}{x\varkappa_n} \Psi_n(0; z, -H_1) + O(R^{-2})$. The support of the function $\eta_2(\theta)$ in the remaining integral ζ_{n2} is a combination of three intervals in which $|\Delta_{n1}|$ is bounded uniformly from below. Using (4) and taking account of the signs of Δ_{n1} in these intervals, we find

$$\zeta_{n2} = \pi \operatorname{Im} \int\limits_{\theta_3}^{0} \Psi_{n2} e^{iR\Delta_n} d\theta + \pi \operatorname{Im} \int\limits_{\theta_4}^{\pi/2} \Psi_{n2} e^{iR\Delta_n} d\theta + \operatorname{Re} \int\limits_{-\pi/2}^{\pi/2} \Psi_{n2} G(R|\Delta_n|) d\theta, \tag{13}$$
$$\theta_3 = \min(\theta_0, \theta_1) \text{ и } \theta_4 = \max(\theta_0, \theta_1).$$

The first integral in (13) differs from zero only for $\gamma < \pi/2$ and has at least one stationary point since $\Delta'_{n\theta}(\theta_1) < 0$ and $\Delta'_{n\theta}(0) > 0$. The formulas for the complete asymptotic expansion of the contributions of the simple, multiple, and almost stationary points are given in [8]. There are no other critical points for this component of (13). In every specific case in which the stationary points can be found, let us denote their total contribution by $S_n(R, \gamma)$. The second component in (13) in the domain $0 < \gamma \leq \pi$ under consideration has no critical points [1], consequently its contribution to the wave field is $O(R^{-\infty})$. The asymptotic of the last component in (13) is derived from (5) and the theorem on integration of asymptotic series [8]. Consequently, as $R \to +\infty$

$$\zeta_{n2} = S_n(R, \gamma) + D_n(R, \gamma), \tag{14}$$

$$D_n(R, \gamma) \sim -\sum_{m=1}^{\infty} (-1)^m R^{-2m} (2m-1)! \int\limits_{-\pi/2}^{\pi/2} \Psi_{n2} \Delta_n^{-2m} d\theta.$$

Thus, if $\gamma \neq \pi/2$ in the far wave field domain ζ_n to $O(R^{-\infty})$ accuracy equals the sum of the contribution of the boundary point, the zeros Δ_n (12), the stationary points, and the series $D_n(R, \gamma)$ (14). The sum of the series in even powers of R from (12) and (14) can be written in the form

$$Z_{n0}(R, \gamma) + Z_{n1}(R, \gamma) + D_n(R, \gamma) \sim -\sum_{m=1}^{\infty} (-1)^m R^{-2m} (2m-1)! \int\limits_{-\pi/2}^{\pi/2} \Psi_n \Delta_n^{-2m} d\theta,$$

where Δ_n^{-2m} should be considered as generalized functions [9].

As $\gamma \to \pi/2$, merger of the zeros θ_0 and θ_1 of the argument of the function G in (3) occurs; consequently, the asymptotic expansions obtained are not uniform in γ, $0 < \gamma_1 \leq \gamma \leq \pi$. Let us calculate the asymptotic ζ_n in the neighborhood of the leading front, the plane $x = 0(\gamma = \pi/2)$. Let ω be a small positive number, $|\theta_1| < \omega$, $V_0 = (-2\omega, 2\omega)$ the neighborhood of the point $\theta = 0$, $\eta_0(\theta)$ an infinitely differentiable function equal to one for $|\theta| \leq \omega$ and zero outside V_0, while $\eta_2(\theta) = 1 - \eta_0(\theta)$. In these notations ζ_n equals the sum

$$\zeta_n = \zeta_{n0} + \zeta_{n2}. \tag{15}$$

Here (11) is valid for ζ_{n0}, while (13) without the first component is valid for ζ_{n2}, and therefore to $O(R^{-\infty})$ accuracy

$$\zeta_{n2} = D_n(R, \gamma). \tag{16}$$

The function Δ_n satisfies the conditions (6.1.20) [8] $\Delta'_{n\theta}(0) = 0$, $\Delta''_{n\theta\theta}(0) = 2\varkappa_n$, $\Delta''_{n\theta\gamma}(0) = -\varkappa_n$ for $\gamma = \pi/2$, consequently [8] the equation $\Delta'_{n\theta} = 0$ has just one solution $\theta_2(\gamma)$ for γ sufficiently close to $\pi/2$ and replacement of the variable $\theta = u(\xi, \gamma)$, is possible for which

$$\xi^2 = \Delta_n(\theta) - \Delta_{n2},\ \Delta_{n2} = \Delta_n(\theta_2),\ \frac{\partial}{\partial\xi} u(0, \gamma) = \sqrt{\frac{2}{\Delta''_{n\theta\theta}(\theta_2)}} \tag{17}$$

and $u(\xi, \gamma)$ is holomorphic in the neighborhood of the point $(0, \pi/2)$. The replacement (17) permits conversion of (11) to the form

$$\zeta_{n0} = \eta_{n1} + \eta_{n2}, \tag{18}$$

$$\eta_{n1} = \pi \operatorname{Im}\left\{e^{iR\Delta_{n2}} \int_{-\infty}^{B} \varphi_n e^{iR\xi^2} d\xi\right\},\quad \eta_{n2} = \operatorname{Re} \int_{-\infty}^{\infty} \varphi_n G[-R(\xi^2 + \Delta_{n2})]\, d\xi,$$

where $B = \operatorname{sign}\left(\frac{\pi}{2} - \gamma\right)\sqrt{-\Delta_{n2}}$; $\varphi_n(\xi) = \Psi_n \eta_0 u'_\xi$, and φ_n is continued to zero outside the domain of definition of $u(\xi, \gamma)$ in ξ. The complete asymptotic expansions of the integrals (18) as $R \to \infty$ are calculated by integration by parts by using a simple technical recipe. Let us demonstrate it in the expansion of the first of the integrals in (18):

$$\int_{-\infty}^{B} \varphi_n(\xi) e^{iR\xi^2} d\xi = \varphi_n(0) \int_{-\infty}^{B} e^{iR\xi^2} d\xi + \int_{-\infty}^{B} \frac{[\varphi_n(\xi) - \varphi_n(0)]}{2iR\xi} de^{iR\xi^2} =$$

$$= \varphi_n(0) \int_{-\infty}^{B} e^{iR\xi^2} d\xi + \frac{\varphi_n(B) - \varphi_n(0)}{2iRB} e^{iRB^2} - \frac{1}{2iR} \int_{-\infty}^{B} P(\varphi_n) e^{iR\xi^2} d\xi,\quad P(\varphi_n) = \frac{d}{d\xi}\left[\frac{\varphi_n(\xi) - \varphi_n(0)}{\xi}\right].$$

Continuation of this procedure yields the asymptotic expansion for the first component in (18)

$$\eta_{n1} \sim \pi \operatorname{Im} \sum_{m=0}^{\infty} (-2iR)^{-m} \left\{ P^m(\varphi_n) \Big|_0 e^{iR\Delta_{n2}} \int_{-\infty}^{B} e^{iR\xi^2} d\xi + \frac{P^m(\varphi_n)}{2iRB}\Big|_0^B \right\}. \tag{19}$$

The integral in (19) is expressed in terms of a special function, the Fresnel integral

$$\int_{-\infty}^{B} e^{iR\xi^2} d\xi = \frac{1}{2}\sqrt{\frac{\pi}{R}}\, \Phi(BR^{1/2}),\quad \Phi(u) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{u} e^{it^2} dt.$$

As a result of the same integration by parts for η_{n2}

we obtain the series

$$\eta_{n2} \sim \sum_{m=0}^{\infty} (-1)^m (2R)^{-2m} \left\{ P^{2m}(\varphi_n)|_0 J_1 - \frac{1}{2R} P^{2m+1}(\varphi_n)|_0 J_2 - \frac{1}{4R^2} \int_{-\infty}^{\infty} P^{2m+2}(\varphi_n) \ln|s_n(\xi)|\, d\xi \right\}; \tag{20}$$

$$J_1 = \operatorname{Re} \int_{-\infty}^{\infty} G[-Rs_n(\xi)]\, d\xi,\ J_2 = \operatorname{Re} \int_{-\infty}^{\infty} F[-Rs_n(\xi)] d\xi,\ s_n(\xi) = \xi^2 + \Delta_{n2}. \tag{21}$$

The integrands in (21) are holomorphic in the domain $0 < \arg \xi < \pi$, and since it follows from (5) that $|G(-Rs_n| = O(|\xi|^{-4})$ and $|F(-Rs_n)| = O(|\xi|^{-2})$ as $|\xi| \to \infty$, the integrals in (21) equal zero. Taking this into account, we rewrite(20):

$$\eta_{n2} \sim \sum_{m=1}^{\infty} (-1)^m (2R)^{-2m} \int_{-\infty}^{\infty} P^{2m}(\varphi_n) \ln|s_n(\xi)|\, d\xi. \tag{22}$$

Formulas (15), (16), (18), (19), and (22) yield the total asymptotic expansion for ζ_n in the neighborhood of the leading front. The principal term of the asymptotic ζ_n as $\gamma \to \pi/2$ has the form

$$\zeta_n = \pi \operatorname{Im}\left\{\Psi_n \sqrt{\frac{\pi}{2R\Delta''_{n\theta\theta}}}\, e^{iR\Delta_n}\Phi(BR^{1/2})|_{\theta=\theta_2}\right\} + \frac{\Psi_n}{RB\sqrt{2\Delta''_{n\theta\theta}}}\Bigg|_{\theta=\theta_2} - \tag{23}$$

$$- \frac{\Psi_n}{\varkappa_n x}\Bigg|_{\theta=0} + O(R^{-3/2}|\Phi(BR^{1/2})|).$$

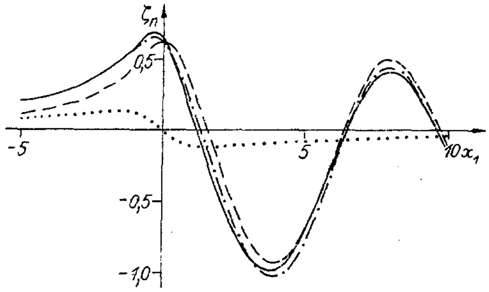

Fig. 1

The representation of the closeness of the principal term of its asymptotic (23) to ζ_n is given in the figure. Computations were performed for a fluid with a constant Brunt–Väisälä frequency; the Boussinesq approximation and "solid cover" condition were used. Values of ζ_n are given in the figure to the accuracy of the factor $\pi\Psi_n$, which is independent of θ in this example [3], $x_1 = x\pi n/H$, $y = H/\pi n$. The solid curve corresponds to ζ_n, computed by means of (3); the dashes correspond to the first term in (23), containing the Fresnel integral; the dots correspond to the sum of the next two components, which are $O(R^{-1})$; and the dash-dot corresponds to the whole principal term of the asymptotic ζ_n. The results of the computations performed show that the asymptotic obtained describes the contribution of the n-th mode in the wave field sufficiently well for $c = c_n$ even at a moderate distance from the wave generator. Taking account of terms of order $O(R^{-1})$ improves the asymptotic estimate (23) substantially.

In conclusion we note that the integral for the η_n contribution of the n-th mode in the wave field formed by a point source [6] diverges for $c = c_n$. However, the displacement field $\eta(x, y, z)$, generated by a system of source-sinks of intensity Q at the points $(-a, 0, -H_1)$ and $(a, 0, -H_1)$, is defined for $c = c_n$ and is related to $\zeta(x, y, z)$ for a dipole by the formula $\eta(x, y, z) = QM^{-1}\int_{-a}^{a}\zeta(x+\xi, y, z)\,d\xi$. The asymptotic for the leading fronts of the waves in the cases $c > c_n$ [4] and $c < c_n$ [6] is expressed in terms of the Airy functions in contrast to (23). Therefore, known asymptotics for the leading fronts are not uniform in c for c close to c_n.

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